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Solution Manual to Mas-Colell
Chapter 2. Consumer Choice

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2.D. COMPETITIVE BUDGETS

EXERCISE 2.D.1.

$\{x \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \le w\}$ where p_i is the price of that consumption good in period i.

EXERCISE 2.D.2.

$\{(x,h) \in \mathbb{R}^2_+ : h \le 24, px + h \le 24\}$.

EXERCISE 2.D.3.

(a) No.

(b) Suppose that X is convex. For any $x, y \in B_{p,w}$, $\alpha x + (1-\alpha)y \in X$ because X is convex. Moreover, $p \cdot (\alpha x + (1-\alpha)y) = \alpha p\dot{x} + (1-\alpha)p \cdot y \le w$. Hence $\alpha x + (1-\alpha)y \in B_{p,w}$.

EXERCISE 2.D.4.

There are two kinks in the graph. The coordinates of the right one is $(16, 8s)$ and the coordinates of the left one is $(\frac{16s'+8s-M}{s'}, M)$. Consider the segment between $(\frac{16s'+8s-M}{s'}, M)$ and

$(24,0)$ and a point on this segment, $\frac{8s'}{M+8s'-8s}(\frac{16s'+8s-M}{s'}, M) + \frac{8s'}{M+8s'-8s}(24,0) = (16, \frac{8Ms'}{M+8s'-8s})$. Since $s' > s$ and $M > 8s$, $\frac{M-8s}{s'} < \frac{M-8s}{s}$. Hence,

$$\frac{8Ms'}{M+8s'-8s} = \frac{8M}{8+\frac{M-8s}{s'}} > \frac{8M}{8+\frac{M-8s}{s}} = 8s.$$

It means that the budget is not convex.

2.E. Demand Functions and Comparative Statics

Exercise 2.E.1.

Since

$$\begin{aligned}
x_1(\alpha p, \alpha w) &= \frac{\alpha p_2}{\alpha p_1 + \alpha p_2 + \alpha p_3} \frac{\alpha w}{\alpha p_1} = \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1} = x_1(p,w) \\
x_2(\alpha p, \alpha w) &= \frac{\alpha p_3}{\alpha p_1 + \alpha p_2 + \alpha p_3} \frac{\alpha w}{\alpha p_2} = \frac{p_3}{p_1 + p_2 + p_3} \frac{w}{p_2} = x_2(p,w) \\
x_3(\alpha p, \alpha w) &= \frac{\alpha \beta p_1}{\alpha p_1 + \alpha p_2 + \alpha p_3} \frac{\alpha w}{\alpha p_3} = \frac{\beta p_1}{p_1 + p_2 + p_3} \frac{w}{p_3} = \beta x_3(p,w),
\end{aligned}$$

it satisfies homogeneous of degree zero if and only if $\beta = 1$. On the other hand, $p \cdot x = \frac{\beta p_1 + p_2 + p_3}{p_1 + p_2 + p_3} w$. It satisfies Walra's law also if and only if $\beta = 1$.

Exercise 2.E.2.

By equation (2.E.4), we have

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} \frac{p_k}{w} + x_k(p,w) \frac{p_k}{w} = 0$$

or

$$\sum_{l=1}^{L} \frac{p_l x_l(p,w)}{w} \frac{\partial x_l(p,w)}{\partial p_k} \frac{p_k}{x_l(p,w)} + \frac{p_k x_k(p,w)}{w} = 0.$$

By replacing them with the notations, we can derive

$$\sum_{l=1}^{L} b_l(p,w) \varepsilon_{lk}(p,w) + b_k(p,w) = 0.$$

Exercise 2.E.3.

By proposition 2.E.1, we have

$$D_p x(p,w) p + D_w x(p,w) w = 0,$$

and

$$p \cdot D_p x(p,w) p + p \cdot D_w x(p,w) w = 0.$$

Besides, by proposition 2.E.3, we have $p \cdot D_w x(p,w) = 1$, so we can infer that

$$p \cdot D_p x(p,w) p = -w.$$

An interpretation is that, when the increment of price is the current price, in other words, the price is doubled, the increment of total expense would be the current wealth.

EXERCISE 2.E.4.

By differentiating both sides of $x(p,\alpha w) - \alpha x(p,w) = 0$ at $\alpha = 1$, we can derive

$$\frac{\partial x(p,w)}{\partial w} w - x(p,w) = 0.$$

In terms of elasticities, we have

$$\varepsilon_{\ell w}(p,w) = \frac{\partial x_\ell(p,w)}{\partial w} \frac{w}{x_\ell(p,w)} = 1.$$

An interpretation is that, since the increment of consumption bundle proportionally equals the increment of the wealth. By the definition of the elasticities, it should be one.

Since the ℓth entry in $D_w x(p,w)$ is $\frac{\partial x_\ell(p,w)}{\partial w} = \frac{x_\ell(p,w)}{w} = x_\ell(p,1)$, $D_w x(p,w)$ only depends on p. The last equity holds because of the homogeneous assumption. Moreover, the Engel curve is a straight line going through $x(p,1)$.

EXERCISE 2.E.5.

Since $x(p,w)$ is homogeneous of degree one with respect to w. We have $\frac{x(p,w)}{w} = x(p,1)$. Moreover, for $k \neq \ell$, $\frac{\partial x_\ell(p,1)}{\partial k} = \frac{1}{w}\frac{\partial x_\ell(p,w)}{\partial p_k} = 0$. Hence, we can infer that $\frac{x_\ell(p,w)}{w} = f_\ell(p_\ell)$ is a function of p_ℓ. By the homogeneity of degree zero ($x_\ell(p,w)$),

$$f_\ell(\alpha p_\ell) = \frac{x_\ell(\alpha p, w)}{w} = \frac{x_\ell(p, \frac{1}{\alpha} w)}{w} = \frac{1}{\alpha}\frac{x_\ell(p,w)}{w} = \frac{1}{\alpha} f_\ell(p_\ell).$$

Furthermore, when $\alpha = \frac{1}{p_\ell}$, $f_\ell(1) = f_\ell(\frac{1}{p_\ell} p_\ell) = p_\ell f_\ell(p_\ell)$. Hence, $f_\ell(p_\ell) = \frac{\alpha_\ell}{p_\ell}$, or so-called homogeneous of degree -1. Therefore, $x_\ell(p,w) = w f_\ell(p_\ell) = \frac{\alpha_\ell w}{p_\ell}$. By Walras' law, we have $w = p \cdot x = \sum p_\ell \frac{\alpha_\ell w}{p_\ell} = w \sum \alpha_\ell$, and we can know that $\sum \alpha_\ell = 1$.

EXERCISE 2.E.6.

We first verify proposition 2.E.1. By differentiating directly, we have

$$\begin{aligned}\frac{\partial x_1(p,w)}{\partial p_1}p_1 =& -(2p_1+p_2+p_3)\frac{p_2}{(p_1+p_2+p_3)^2}\frac{w}{p_1}\\ \frac{\partial x_1(p,w)}{\partial p_2}p_2 =& \frac{(p_1+p_3)p_2}{(p_1+p_2+p_3)^2}\frac{w}{p_1}\\ \frac{\partial x_1(p,w)}{\partial p_3}p_3 =& \frac{-p_2p_3}{(p_1+p_2+p_3)^2}\frac{w}{p_1}\\ \frac{\partial x_1(p,w)}{\partial w}w =& \frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1}.\end{aligned}$$

Hence, we have

$$\sum_{k=1}^{3}\frac{\partial x_1(p,w)}{\partial p_k}p_k+\frac{\partial x_1(p,w)}{\partial w}w=0.$$

By the same logic, verifying the statement still hold for $x_2(p,w)$ and $x_3(p,w)$ would be easy. To verify proposition 2.E.3, we derive

$$\begin{aligned}\frac{\partial x_1(p,w)}{\partial w} =& \frac{p_2}{p_1+p_2+p_3}\frac{1}{p_1},\\ \frac{\partial x_2(p,w)}{\partial w} =& \frac{p_3}{p_1+p_2+p_3}\frac{1}{p_2},\\ \frac{\partial x_3(p,w)}{\partial w} =& \frac{p_1}{p_1+p_2+p_3}\frac{1}{p_3}.\end{aligned}$$

Hence, we can infer that

$$\sum_{\ell=1}^{3}p_\ell\frac{\partial x_\ell(p,w)}{\partial w}=1.$$

EXERCISE 2.E.7.

By Walras' law, we have $p_1x_1(p,w)+p_2x_2(p,w)=w$. Imposing the function form of good , we can infer that $x_2(p,w)=\frac{(1-\alpha)w}{p_2}$. It is homogeneous of degree zero.

EXERCISE 2.E.8.

We first rewrite $x_\ell(p,w)$ into $x_\ell(e^{\ln p_1},\dots,e^{\ln p_L},e^{\ln w})$. Then we can derive

$$\begin{aligned}&\frac{d\ln(x_\ell(p,w))}{d\ln(p_k)}\\ =&\frac{1}{x_\ell(p,w)}\frac{\partial x_\ell(p,w)}{\partial p_k}e^{\ln p_k}\\ =&\frac{\partial x_\ell(p,w)}{\partial p_k}\frac{p_k}{x_\ell(p,w)}\\ =&\varepsilon_{\ell k}(p,w).\end{aligned}$$

By the same logic, we can derive

$$
\begin{aligned}
&\frac{d\ln(x_\ell(p,w))}{d\ln(w)}\\
=&\frac{1}{x_\ell(p,w)}\frac{\partial x_\ell(p,w)}{\partial w}e^{\ln w}\\
=&\frac{\partial x_\ell(p,w)}{\partial w}\frac{w}{x_\ell(p,w)}\\
=&\varepsilon_{\ell w}(p,w).
\end{aligned}
$$

In the estimation, since $\frac{d\ln(x_\ell(p,w))}{d\ln(p_1)}=\alpha_1$, $\frac{d\ln(x_\ell(p,w))}{d\ln(p_2)}=\alpha_2$, $\frac{d\ln(x_\ell(p,w))}{d\ln(w)}=\gamma$, it provides us an estimation of each elasticity.

2.F. The Weak Axiom of Revealed Preference and the Law of Demand

Exercise 2.F.1.

We first assume that $p\cdot x(p',w')\le w$ and $p'\cdot x(p,w)\le w'$. In the term of chapter 1, $B=\{x\in\mathbb{R}^L_+ : p\cdot x\le w\}$ and $B'=\{x\in\mathbb{R}^L_+ : p'\cdot x\le w'\}$. By definition 1.C.1, since $x(p,w),x(p',w')\in B$, $x(p,w),x(p',w')\in B'$, and the single-value assumption, we require $x(p,w)=x(p',w')$. Thus, if we assume $p\cdot x(p',w')\le w$ and $x(p,w)\ne x(p',w')$, $p'\cdot x(p,w)\le w'$ would not hold. We can therefore infer that $p'\cdot x(p,w)>w'$.

Exercise 2.F.2.

To prove that x^3 is revealed preferred to x^2, it is sufficient to show that x^2 is feasible under price p^3. Since $p^3\cdot x^2=8$, it is feasible. Similarly, $x^1\cdot p^2=8$ implies that x^1 is feasible under p^2. It implies that x^2 is revealed preferred to x^1. $p^1\cdot x^3=8$ implies that x^1 is revealed preferred to x^3.

Exercise 2.F.3.

Denote the quantity of good 2 in year 2 by x.

(a) If the consumption bundle of year 1 is feasible in year 2 and the consumption bundle of year 2 us feasible in year 1, then the behavior is inconsistent because two bundles are different. Hence, it is inconsistent when the following equations both hold:

$$
\begin{aligned}
12000+100x&\le 20000\\
18000&\le 12000+80x.
\end{aligned}
$$

Both equations hold when $x\in[75,80]$.

(b) The bundle in year 1 is revealed preferred if the bundle in year 1 is infeasible in year 2 and the bundle in year 2 is feasible in year 1. That is,

$$12000 + 100x \leq 20000$$
$$18000 > 12000 + 80x.$$

Both equations hold when $x < 75$.

(c) Similarly, the bundle in year 2 is revealed preferred if the bundle in year 1 is feasible in year 2 and the bundle in year 2 is infeasible in year 1. That is,

$$12000 + 100x > 20000$$
$$18000 \leq 12000 + 80x.$$

Both equations hold when $x > 80$.

(d) Since the three intervals form a partition of real number, a situation with any quantity of good 2 in year 2 can be justify.

(e) Since the quantity of good 1 is more in year 2, it is sufficient t prove that the wealth is less in year 2. It means that the consumption bundle in year 1 is more expensive than the bundle in year 2 under two prices. Thus,

$$20000 > 12000 + 100x$$
$$18000 > 12000 + 80x.$$

We an conclude that good 1 is an inferior good when $x < 75$.

(f) There are two possible cases to conclude that good 2 is inferior. First, if $x > 100$, we have to prove that the wealth in year 2 is less. However, by (e), the wealth in year 2 is less when $x < 75$. It never holds when $x > 100$.

Second, if $x < 100$, we are going to prove that the wealth in year 2 in more. Similarly, it means that the consumption bundle is more expensive in year 2 under two prices. Thus,

$$20000 < 12000 + 100x$$
$$18000 < 12000 + 80x.$$

We an conclude that good 2 is an inferior good when $x \in (80, 100)$.

EXERCISE 2.F.4.

(a) If $L_Q < 1$, then $p^0 \cdot x^1 < p^0 \cdot x^0$. It means x^1 is feasible under price p^0, and the optimal consumption bundle under p^0 is x^0. Hence, x^0 is revealed preferred to x^1.

(b) $P_Q > 1$ implies $p^1 \cdot x^0 < p^1 \cdot x^1$. Similarly, It means that x^1 is revealed preferred to x^0.

(c) Since $E_Q = w^1 / w^0$, the relationship between E_Q and 1 only reflects the change of the wealth. It would not revealed any information of preference.

Exercise 2.F.5.

Denote the compensated wealth $p' \cdot x(p,w)$ by w'. We should prove $(p'-p)\cdot[x(p',w) - x(p,w)] \le 0$. The prove goes as follows:

$$
\begin{aligned}
&(p'-p)\cdot[x(p',w)-x(p,w)]\\
=&(p'-p)\cdot[\frac{w}{w'}(x(p',w')-x(p,w)) + \frac{w}{w'}x(p,w) - x(p,w)]\\
=&\frac{w}{w'}(p'-p)\cdot[x(p',w')-x(p,w)] + (\frac{w}{w'}-1)(p'-p)\cdot x(p,w)\\
\le&(\frac{w}{w'}-1)(w'-w)\\
=&-\frac{1}{w'}(w'-w)^2\\
\le&\,0
\end{aligned}
$$

The first inequality holds by the original law of demand.

The proof of the infinitesimal version goes as follows. $S(p,w)$ defined as $D_p x(p,w) + D_w x(p,w)x(p,w)^T$ has been proved to be negative semidefinite. In exercise 2.E.4, we have argued that $D_w x(p,w) = x(p,1)$ when $x(p,w)$ is homogeneous of degree one with respect to w. Hence, $D_w x(p,w)x(p,w)^T = \frac{1}{w}x(p,w)x(p,w)^T$ is positive semidefinite. As a result, $D_p x(p,w)$ has to be negative semidefinite.

Exercise 2.F.6.

First, the necessity is trivial. The statement should hold under the same wealth. Then, since $x(p,w)$ is homogeneous of degree zero, $x(p',w') = x(\frac{w}{w'}p',w)$ for any w'. Suppose the statement in the exercise holds. Give (p,w) and (p',w'), the two statements are equivalent if we replace $x(p',w')$ with $x(\frac{w}{w'}p',w)$. Then the original statement would hold.

Exercise 2.F.7.

In matrix notation,

$$
\begin{aligned}
&p\cdot S(p,w)\\
=&p\cdot D_p x(p,w) + p\cdot D_w x(p,w)x(p,w)^T\\
=&p\cdot D_p x(p,w) + x(p,w)^T\\
=&0^T.
\end{aligned}
$$

The second and the third equality hold because of proposition 2.E.3 and 2.E.2 respectively.

On the other hand,

$$
\begin{aligned}
&S(p,w)p\\
=&D_p x(p,w)p + p\cdot D_w x(p,w)x(p,w)^T p\\
=&D_p x(p,w)p + p\cdot D_w x(p,w)w\\
=&0
\end{aligned}
$$

The second and the third equality hold because of the Walras' law and proposition 2.E.1 respectively.

EXERCISE 2.F.8.

By definition,

$$\begin{aligned}
&\hat{s}_{\ell k}(p,w)\\
=&\frac{p_k}{x_\ell(p,w)}\frac{\partial x_\ell(p,w)}{\partial p_k}+\frac{p_k}{x_\ell(p,w)}\frac{\partial x_\ell(p,w)}{\partial w}x_k(p,w)\\
=&\frac{\partial x_\ell(p,w)}{\partial p_k}\frac{p_k}{x_\ell(p,w)}+\frac{\partial x_\ell(p,w)}{\partial w}\frac{w}{x_\ell(p,w)}\frac{p_k x_k(p,w)}{w}\\
=&\varepsilon_{\ell k}(p,w)+\varepsilon_{\ell w}(p,w)b_k(p,w)
\end{aligned}$$

EXERCISE 2.F.9.

(a) Since $x^T A^T x = (x^T A^T x)^T = x^T A x$, then $x^T(A+A^T)x = x^T Ax + x^T A^T x = x^T 2Ax$. Therefore, A is negative (semi)definite if and only if $A+A^T$ is negative (semi)definite. If $A = \left[\begin{smallmatrix}-1 & 2\\ 0 & -1\end{smallmatrix}\right]$, $(-1)A_{11} = -1$ and $(-1)^2 A_{22} = -1$. The determinant holds. However,

$$\begin{bmatrix}1\\1\end{bmatrix}^T\begin{bmatrix}1 & 3\\1 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=0$$

It means that A is not negative semidefinite.

(b) By proposition 2.F.3, $p\cdot S(p,w)=0$ means $p_1 s_{11}(p,w)+p_2 s_{21}(p,w)=0$ and $p_1 s_{21}(p,w)+p_2 s_{22}(p,w)=0$. $S(p,w)p=0$ means $p_1 s_{11}(p,w)+p_2 s_{12}(p,w)=0$ and $p_1 s_{21}(p,w)+p_2 s_{22}(p,w)=0$. Hence, we have

$$s_{12}(p,w)=s_{21}(p,w)=-\frac{p_1}{p_2}s_{11}(p,w)$$

$$s_{22}(p,w)=(\frac{p_1}{p_2})^2 s_{11}(p,w)$$

Then for any vector $x=[x_1 x_2]^T$,

$$\begin{aligned}
x^T S(p,w)x &= x_1^2 s_{11}(p,w)+x_1x_2[s_{21}(p,w)+s_{12}(p,w)]+x_2^2 s_{22}(p,w)\\
&= s_{11}(p,w)[x_1^2-2\frac{p_1}{p_2}x_1x_2+(\frac{p_1}{p_2})^2x_2^2]\\
&= s_{11}(p,w)(x_1-\frac{p_1}{p_2}x^2)^2\\
&= s_{22}(p,w)(x_2-\frac{p_2}{p_1}x^1)^2
\end{aligned}$$

Therefore, since $S(p,w)$ is of rank 1, any vector is not parallel with the price vector should belong its null space. Hence, both $s_{11}(p,w)$ and $s_{22}(p,w)$ cannot be zero. Moreover, to satisfy negative semidefinite, both $s_{11}(p,w)$ and $s_{22}(p,w)$ should be negative.

EXERCISE 2.F.10.

(a) With plugging the numbers directly, we can derive

$$S(p,w)=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 1 & 0 & -1\end{bmatrix}$$

. For any vector $x=\begin{bmatrix}a & b & c\end{bmatrix}^T$, we have

$$\begin{aligned}&\begin{bmatrix}a\\ b\\ c\end{bmatrix}^T\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}\\ =&-a^2-b^2-c^2+ab+ac+bc\\ =&-\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]\\ \le&0,\end{aligned}$$

and it is negative semidefinite.

(b) Let $p=(1,1,\varepsilon)$ and $w=1$. To simplify the calculation, let $\hat{S}(p,w)$ be the 2×2 submatrix of $S(p,w)$ obtained by deleting the last row and column. With plugging the numbers directly, we have

$$\hat{S}(p,w)=\frac{1}{(2+\varepsilon)^2}\begin{bmatrix}-2-\varepsilon & 1+2\varepsilon\\ 0 & -3\varepsilon.\end{bmatrix}$$

Hence,

$$\begin{bmatrix}1\\ 4\\ 0\end{bmatrix}^T S(p,w)\begin{bmatrix}1\\ 4\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}^T\hat{S}(p,w)\begin{bmatrix}1\\ 4\end{bmatrix}=\frac{2-41\varepsilon}{(2+\varepsilon)^2}.$$

It might be positive if ε is small enough. Thus, $S(p,w)$ is not negative semidefinite. In Exercise 2.E.1, we have argued that this demand function satisfy homogeneous of degree zero and Walras' law when $\beta=1$. However, the demand function does not satisfy proposition 2.F.2. Hence, it dose not satisfy the weak axiom.

EXERCISE 2.F.11.

Let $p=(p_1,p_2)$. By proposition 2.F.3, we have $p\cdot S(p,w)=S(p,w)p=0$. When $L=2$, $p\cdot S(p,w)=0$ implies $p_1 s_{11}(p,w)+p_2 s_{21}(p,w)=0$; meanwhile, $S(p,w)p=0$ implies $p_1 s_{11}(p,w)+p_2 s_{12}(p,w)=0$. Thus, $s_{12}(p,w)=s_{21}(p,w)=-\frac{p_1}{p_2}s_{11}(p,w)$. It means that $S(p,w)$ is symmetric when $L=2$.

EXERCISE 2.F.12.

A demand function is single-value. Therefore, we can assume that, in each $B_{p,w}$, there exists a consumption bundle is strictly preferred to others. Thus, if $x(p,w)$ is generated by a rational preference $\succ$, $x \succ y$ and $y \succ x$ cannot hold in the same time. $p \cdot x(p',w') \leq w$ implies $x(p,w) \succ x(p',w')$. Since the preference is rational, $x(p,w)$ must be infeasible in $B_{p,w}$.

EXERCISE 2.F.13.

Exercise correction:

- In (b), equation $(*)$, $p \cdot x > w$ should be $p' \cdot x > w'$
- In the last part of (c), $x' \in x(p,w)$ should be $x' \notin x(p,w)$

(a) We rewrite the definition of the weak axiom based on Definition 1.C.1:

If for some (p,w) with $x, y \in B_{p,w}$ we have $x \in x(p,w)$, then for any (p',w') with $x, y \in B_{p',w'}$ and $y \in x(p',w')$, we must also have $x \in x(p',w')$.

(b) Since the demand function satisfies Walras' law, $p \cdot x' < w$ implies $x' \in B_{p,w}$ but $x' \notin x(p,w)$. Hence, we have $x, x' \in B_{p,w}$, $x' \in B_{p',w'}$, and $x' \in x(p',w')$. Suppose that $x \in B_{p',w'}$, by the definition above, we can infer that $x' \in x(p,w)$, a contradiction. Hence, we can conclude that $x \notin B_{p',w'}$. That is, $p' \cdot x > w$.

(c) By definition,

$$\begin{aligned}
&(p'-p)\cdot(x'-x)\\
=&p'\cdot x' + p\cdot x - p\cdot x' - p'\cdot x\\
=&w' + w - p\cdot x - w'\\
=&w - p\cdot x'.
\end{aligned}$$

If $x' \in x(p,w)$, Walras' law implies $p \cdot x' = w$ and therefore $(p'-p)\cdot(x'-x) = 0$. Otherwise, if $x' \notin x(p,w)$, weak axiom implies $x' \notin B_{p,w}$. It means that $p \cdot x' > w$, and hence $(p'-p)\cdot(x'-x) < 0$.

(d) In the proposition 2.F.1, it is shown that the violation of the weak axiom must lead to a violation under compensated price change.Therefore, in order to verify the assertion , it is sufficient to show that the generalized weak axiom holds for all compensated price change.

Given (p,w), p', and $x \in x(p,w)$, let $w' = p' \cdot x$. For any $x' \in x(p',w')$, suppose that $x' \in B_{p,w}$. It means that $p \cdot x' \leq w$. As a result, $(p'-p)\cdot(x'-x) = w - p\cdot x' \leq 0$. If the inequality in (c) holds, we require $(p'-p)\cdot(x'-x) = 0$. Moreover, the equality implies $x' \in x(p,w)$, and the weak axiom holds.

Exercise 2.F.14.

For $\alpha > 0$, let $p' = \alpha p$ and $w' = \alpha w$. Since $p' \cdot x(p', w') \leq w'$, it implies $p \cdot x(\alpha p, \alpha w) \leq w$. If $x(\alpha p, \alpha w) \neq x(p, w)$, by weak axiom, it implies $p' \cdot x(p, w) > w'$. It is equivalent to $p \cdot x(p, w) > w$, a contradiction. Hence, we can infer that $x(\alpha p, \alpha w) = x(p, w)$.

Exercise 2.F.15.

By Walras' law, $x_3(p, w) = \frac{w + p_1^2 + p_2^2 - p_1 p_2}{p_3}$. Following Theorem M.D.4 (iii), let $\hat{S}(p, w)$ be the submatrix with deleting the last row and column. With plugging the numbers directly, we have

$$\hat{S}(p, w) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

For any nonzero vector $x = (a, b)$,

$$\begin{aligned}
& x^T \hat{S}(p, w) x \\
=& -a^2 + ab - b^2 \\
=& -(a - \frac{1}{2} b)^2 - \frac{3}{4} b^2 \\
\leq& -\frac{3}{4} b^2 \\
<& 0
\end{aligned}$$

By the theorem, negative definite $\hat{S}(p, w)$ implies that $S(p, w)$ is negative definite for any vector not proposition to p. Moreover, $S(p, w)$ is not symmetric since $\hat{S}(p, w)$ is not.

Exercise 2.F.16.

(a) The demand function is homogeneous of degree zero because for all $\alpha > 0$

$$\begin{aligned}
x_1(\alpha p, \alpha w) &= \frac{\alpha p_2}{\alpha p_3} = \frac{p_2}{p_3} = x_1(p, w), \\
x_2(\alpha p, \alpha w) &= -\frac{\alpha p_1}{\alpha p_3} = -\frac{p_1}{p_3} = x_2(p, w), \\
x_3(\alpha p, \alpha w) &= \frac{\alpha w}{\alpha p_3} = \frac{w}{p_3} = x_3(p, w).
\end{aligned}$$

Besides, since $p \cdot x = p_1 \frac{p_2}{p_3} - p_2 \frac{p_1}{p_3} + p_3 \frac{w}{p_3} = w$, it satisfies Walras' law.

(b) Let $p = (1, 2, 1), w = 1$ and $p' = (1, 1, 1), w' = 2$. Then $x(p, w) = (2, -1, 1)$ and $x(p', w') = (1, -1, 2)$. However, $p \cdot x(p', w') = 1 \leq w$ and $p' \cdot x(p, w) = 2 \leq w'$, a contradiction.

(c) With plugging the numbers directly, we have

$$S(p, w) = \frac{1}{p_3^2} \begin{bmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{bmatrix}.$$

Since $S(p,w)$ is an antisymmetric matrix, for all x, $x^T S(p,w)x = \frac{1}{2}x^T(S(p,w)+S(p,w)^T)x = 0$.

Remark: In this case, the demand function violate the weak axiom, but its substitution matrix is still negative semidefinite.

EXERCISE 2.F.17.

(a) For $k = 1,\dots,L$ and $\alpha > 0$,

$$x_k(\alpha p, \alpha w) = \frac{\alpha w}{\sum_{\ell=1}^L \alpha p_\ell} = \frac{\alpha w}{\alpha \sum_{\ell=1}^L p_\ell} = \frac{w}{\sum_{\ell=1}^L p_\ell} x_k(p,w).$$

It means that the demand function is homogeneous of degree of zero.

(b)

$$\begin{aligned} p \cdot x &= \sum_{k=1}^L p_k x_k \\ &= \sum_{k=1}^L p_k \frac{w}{\sum_{\ell=1}^L p_\ell} \\ &= w \frac{\sum_{k=1}^L p_k}{\sum_{\ell=1}^L p_\ell} \\ &= w \end{aligned}$$

It means that the demand function satisfies Walras' law.

(c) If $p \cdot x(p', w') \le w$, $\frac{w' \sum_{\ell=1}^L p_\ell}{\sum_{\ell=1}^L p'_\ell} \le w$. Moreover, suppose that $x(p', w') \ne x(p,w)$ which means $\frac{w'}{\sum_{\ell=1}^L p'_\ell} \ne \frac{w}{\sum_{\ell=1}^L p_\ell}$. We can infer that $\frac{w' \sum_{\ell=1}^L p_\ell}{\sum_{\ell=1}^L p'_\ell} < w$, and therefore $\sum_{\ell=1}^L p'_\ell \frac{w}{\sum_{\ell=1}^L p_\ell} > w'$. Hence, $p' \cdot x(p,w) > w'$, and the weak axiom holds.

(d) For all ℓ, k, $\frac{\partial x_\ell(p,w)}{\partial p_k} = -\frac{w}{(\sum_{\ell=1}^L p_\ell)^2}$ and $\frac{\partial x_\ell(p,w)}{\partial w} = \frac{1}{\sum_{\ell=1}^L p_\ell}$. Therefore, for all ℓ, k, $s_{\ell k}(p,w) = 0$, and $S(p,w)$ is a zero matrix. Moreover, it is negative semidefinite, symmetric, but it is not negative definite.