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Solution Manual to Mas-Colell

Chapter 1. Preference and Choice

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1.A. Preference Relations

Exercise 1.A.1.

Suppose that $x \succ y \succsim z$. The condition can be separated into $x \succsim y$, $y \succsim z$ without $z \succsim y$. By the first two conditions and transitivity, $x \succsim z$ holds. Suppose that $z \succsim x$. By transitivity again and $x \succsim y$, it implies $z \succsim y$, a contradiction.

Exercise 1.A.2.

(i) *irreflexive of* $\succ$: $x \succ x$ implies that $x \succsim x$ dose not hold. It violates the completeness (or reflexive) of $\succsim$.

transitive of $\succ$: $x \succ y$ and $y \succ z$ respectively imply $x \succsim y$ and $y \succsim z$. By transitivity, we have $x \succsim z$. Further, if $z \succsim x$, $y \succsim z$ and transitivity implies $y \succsim x$, a contradiction.

(ii) *reflexive of* $\sim$: $x \succsim x$ implies $x \sim x$.

transitive of $\sim$: $x \sim y$ and $y \sim z$ respectively imply $x \succsim y$, $y \succsim x$, $y \succsim z$, and $z \succsim y$. They imply $x \succsim z$ and $z \succsim x$ by transitivity.

symmetric of $\sim$: $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x \Leftrightarrow y \sim x$

EXERCISE 1.A.3.

By definition, f is a strictly increasing function whenever $a > b$ implies $f(a) > f(b)$. It implies $a \geq b \Leftrightarrow f(a) \geq f(b)$. Suppose that u represents $\succsim$. Thus,

$$x \succsim y \Leftrightarrow u(x) \geq u(y) \Leftrightarrow f(u(x)) \geq f(u(y).)$$

And $v(\cdot) = f(u(\cdot))$ also represents $\succsim$.

EXERCISE 1.A.4.

$x \succsim y$ implies $y \nsucc x$. By assumption, it implies $u(x) \geq u(y)$. Conversely, if $u(x) \geq u(y)$, then one of $u(x) = u(y)$ and $u(x) > u(y)$ holds, which implies one of $x \sim y$ and $x \succ y$ holds. It is as same as $x \succsim y$.

EXERCISE 1.A.5.

We are going to construct a utility function $u : X \to (0,1)$ represents $\succsim$, a rational preference relation. Since X is finite, we assume $X = \{x_1, \ldots, x_n\}, n = |X| < \infty$ and define $X_i = \{x_1, \ldots, x_i\}$. First, We let $u(x_1) = \frac{1}{2}$. Then for any $i \geq 2$, define $U_i = \{x_j \in X_{i-1} : x_j \succsim x_i\}$ and $L_= \{x_j \in X_{i-1} : x_j \precsim x_i\}$. By transitivity of $\succsim$, $x_j \succsim x_k$ for all $x_j \in U_i$ and all $x_k \in L_i$. Define $u(S) = \{u(x) : x \in S\}$. Then there are several cases:

(a) $U_i = \emptyset$:
Assign $u(x_i) = \frac{\max u(L_i)+1}{2}$ so that $u(x_i) > u(x_j)$ for all $x_j \in L_i$.

(b) $U_i \neq \emptyset$, $L_i \neq \emptyset$ and $U_i \cap L_i = \emptyset$:
$U_i \cap L_i = \emptyset$ implies that $x_i \succ x_j$ for all $x_j \in L_i$ and $x_i \prec x_k$ for all $x_k \in U_i$. It also implies $\min u(U_i) > \max u(L_i)$. Then we assign $u(x_i) = \frac{\min u(U_i)+\max u(L_i)}{2}$.

(c) $U_i \cap L_i \neq \emptyset$: By definition, $x_i \sim x^*$ for all $x^* \in U_i \cap L_i$. Then we can let $u(x_i) = u(x^*)$.

(d) $L_i = \emptyset$:
Similar to the first case, we assign $u(x_i) = \frac{\min u(U_i)+0}{2}$.

By the same logic, we can further extend this exercise. The utility exists as long as X is at most countable infinite.

1.B. CHOICE RULES

EXERCISE 1.B.1.

Suppose $y \in C(\{x, y, z\})$. By weak axiom, $x, y \in \{x, y, z\}$ and $y \in C(\{x, y, z\})$ implies $y \in C(\{x, y\})$ whenever $x \in C(\{x, y\})$, a contradiction. Since $y \notin C(\{x, y, z\})$ and $C(\{x, y, z\})$ is nonempty, $C(\{x, y, z\})$ must equal one of $\{x\}$, $\{z\}$, and $\{x, z\}$.

Exercise 1.B.2.

Suppose weak axiom holds. Assume $B, B' \in \mathscr{B}, x, y \in B, x, y \in B', x \in C(B), y \in C(B)$. By weak axiom, we have $x \in C(B')$ and $y \in C(B)$. Then the statement in exercise holds.

Conversely, suppose the statement holds. Assume $x, y \in B, x, y \in B', x \in C(B), y \in C(B')$. We have $\{x, y\} \subset C(B)$ and $\{x, y\} \subset C(B')$. The weak axiom holds.

Exercise 1.B.3.

(a) Suppose $(\mathscr{B}, C(\cdot))$ satisfy weak axiom. If $x \succ^* y$, there is some $B \in \mathscr{B}$ such that $x, y \in B, x \in C(B)$, and $y \notin C(B)$. By $x \in C(B)$, we have $x \succsim^* y$. Suppose $y \succsim^* x$. It implies there is some $B' \supset \{x, y\}$ such that $y \in C(B')$. However, by weak axiom, we should have $y \in C(B)$, but it contradicts the assumption. Hence, we have $x \succ^{**} y$.

Conversely, Suppose $x \succ^{**} y$. We have $x \succsim^* y$ but not $y \succsim^* x$. It means there exists $B \in \mathscr{B}$ such that $x, y \in B$ and $C(B)$, and for all $B' \supset \{x, y\}$, $y \notin C(B')$. So we have $x \succ^* y$ and we can conclude with $x \succ^* y \Leftrightarrow x \succ^{**} y$.

The weak axiom is needed for deriving the first part. In other words, if weak axiom is absent, we can only deduce $x \succ^{**} y \Rightarrow x \succ^* y$.

To construct a counterexample, let $\mathscr{B} = (\{x, y\}, \{x, y, z\})$, $C(\{x, y\}) = \{x\}$, $C(\{x, y, z\}) = \{y\}$. This choice rule obviously violates the weak axiom. We have $x \succ^* y$, but $x \not\succ^{**} y$.

(b) $\succ^*$ may not be transitive even though the choice rule satisfy the weak axiom. It is sensitive to $\mathscr{B}$. If $\mathscr{B} = (\{x, y\}, \{y, z\})$, $C(\{x, y\}) = x$, and $C(\{y, z\}) = \{z\}$. However, there is no $B \in \mathscr{B}$ such that $x, z \in B$, so $x \not\succ^* z$.

(c) If $x \succ^* y$ and $y \succ^* z$, then we have $B_1, B_2 \in \mathscr{B}$ such that $x, y \in B_1, y, z \in B_2, x \in C(B_1), y \notin C(B_1), y \in C(B_2), z \notin C(B_2)$. By assumption, $\{x, y, z\} \in \mathscr{B}$ By weak axiom, if $y \in C(\{x, y, z\})$, we must have $\{x, y\} \subset C(B_1)$ by Exercise 1.C.2 and $x \in C(B_1)$, a contraction. If $z \in C(\{x, y, z\})$, we have $\{y, z\} \subset C(B_2)$ by Exercise 1.C.2 and $y \in C(B_2)$, a contraction. Since $C(x, y, z)$ is nonempty, we should have $\{x\} = C(\{x, y, z\})$ and we can conclude with $x \succ^* z$.

1.C. The relationship between Preference Relation and Choice Rules

Exercise 1.C.1.

If $\mathscr{B} = \{x, y, z\}$ and $C(\{x, y, z\}) = \{x\}$, both $\succsim_1$ with $x \succ_1 y \succ_1 z$ and $\succsim_2$ with $x \succ_2 z \succ_2 y$ can rationalize $C(\cdot)$.

Exercise 1.C.2.

By exercise 1.B.5, if X is finite, there exist $u(\cdot)$ representing $\succsim$. Moreover, since $\{u(x) : x \in B\}$ is also finite for all $B \subset X$, and there is some $x_B \in B$ such that $u(x) \geq u(y)$ for all $y \in B$. Hence, $x_B \in C(B)$ for all $B \in \mathscr{B}$.

Exercise 1.C.3.

We assume that the weak axiom holds. Suppose that $x \in C(\{x, y, z\})$. By exercise 1.C.2, we must have $\{x, z\} \subset C(\{x, z\})$, a contradiction. Similarly, if $y \in C(\{x, y, z\})$, it implies $\{x, y\} \subset C(\{x, y\})$, a contradiction. Finally, $z \in C(\{x, y, z\})$, it implies $\{y, z\} \subset C(\{y, z\})$, a contradiction.

However, $C(\{x, y, z\})$ should be nonempty. Hence, it must violate the weak axiom.

Exercise 1.C.4.

Suppose there is a preference relation $\succsim$ rationalize $(\mathscr{B}, C(\cdot))$.

For any $x \in C(C(B_1) \cup C(B_2))$, $x \succsim y$ for all $y \in C(B_1)$ and $x \succsim z$ for all $z \in C(B_1)$. By definition, $y \succsim y'$ for all $y' \in B_1$ and $z \succsim z'$ for all $z' \in B_2$. Since $\succsim$ is transitive, $x \succsim x'$ for all $x' \in B_1 \cup B_2$. We have $x \in C(B_1 \cup B_2)$ and then $C(C(B_1) \cup C(B_2)) \subset C(B_1 \cup B_2)$.

For $x \in C(B_1 \cup B_2)$, $x \succsim x'$ for all $x' \in B_1 \cup B_2$. Hence, $x \succsim y$ for all $y \in B_1$ and $x \succsim z$ for all $z \in B_2$ and then $x \in C(B_1) \cup C(B_2)$. Furthermore, $x \succsim y'$ for all $y' \in C(B_1)$ and $x \succsim z'$ for all $z' \in C(B_2)$, and then $x \in C(C(B_1) \cup C(B_2))$. Therefore, $C(B_1 \cup B_2) \subset C(C(B_1) \cup C(B_2))$.

As a result, we have $C(B_1 \cup B_2) = C(C(B_1) \cup C(B_2))$.

Exercise 1.C.5.

We define $\succsim_i$ with

- $x \succ_1 y \succ_1 z$
- $x \succ_2 z \succ_2 y$
- $y \succ_3 x \succ_3 z$
- $y \succ_4 z \succ_4 x$
- $z \succ_5 x \succ_5 y$
- $z \succ_6 y \succ_6 x$

For convenience, we let $(p_1, \ldots, p_6)$ which p_i is the probability of $\succ_i$ for $i = 1 \ldots 6$.

(a) If $p_i = \frac{1}{6}$ for all i, then $C_x(\{x, y\}) = p_1 + p_2 + p_5 = \frac{1}{2}$ and other number can be reached similarly.

(b) If this stochastic choice function is rationalizable, then we can infer that $p_6 = p_1 + \frac{1}{2}$, $p_2 = p_4 + \frac{1}{2}$, and $p_3 = p_5 + \frac{1}{2}$. It implies $p_2 + p_3 + p_6 \geq \frac{3}{2}$ which is impossible.

(c) In this case, we require $\{p_i\}$ satisfy $p_2 + P_3 + p_6 = 2 - 3\alpha$. Hence, any $\alpha \notin [\frac{1}{3}, \frac{2}{3}]$ would lead to a contradiction. However, when $\alpha = \frac{1}{3}$, it can be rationalize with $p + 2 = p_3 = p_6 = \frac{1}{3}$. Similarly, if $\alpha = \frac{2}{3}$, it can be done in a symmetric way.

In conclusion, this kind of stochastic choice function is rationalizable only when $\alpha \in [\frac{1}{3}, \frac{2}{3}]$.